Mathematical Analysis1

third level

First course

Mathematics department

College of Education for Pure Sciences

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Axioms of real numbers

1. The axioms arithmetics

2. The axioms of ordered
3. The complete Axioms

* Let R be a real number and $a, b, c \in R$. Then

A_1 : $\forall a, b, c \in R \;\; a + (b + c) = (a + b) + c.$

A_2 : $a + b = b + a$

A_3 : $for\ any\ a \in R, \exists!\ element\ 0 \in R \;\; s.t$

$a + (-a) = -a + a = 0$

A_4 : $Ther\ exists\ an\ element\ 0 \in R\ , S.t$

$a + 0 = 0 + a = a$

Then $(R, +)$ is a commutative group.

A_5 : $a.(b.c) = (a.b).c$

A_6 : $a.b = b.a$

A_7 : $\exists!$ Element in $R(1 \in R\ )\ s.t\ a.1 = 1.a = a$

A_8 : $\forall a \in R\ , \exists!\ a^{-1} \in R\ , s.t\ a.a^{-1} = a^{-1}.a = 1$

Form $A_5 \to A_8$. $(R, .)$ commutive ring

A_9 : $a.(b + c) = (a.b) + (a.c)$

$A_1 \to A_9 \;\; (R, +, .)$ Is a field

Def:

* Subtraction $\quad a - b = a + (-b)\ , \forall a, b \in R$
* Division $\quad a \div b = a.b^{-1} \ni b \neq 0$

The Axioms of order:

A_{10}: $a \leq b\ or\ b \leq a$

A_{11}: $a \leq b\ and\ b \leq c \to a = b$

A_{12}: $a \leq b\ and\ b \leq c \to a \leq c$

A_{13}: $a \leq b\ , c \in R \to a + c \leq b + c$

A_{14}: $a \leq b\ , c\ is\ not\ negative \to a.c < -b.c$

$A_1 \to A_{14}\ , (R, +\ , .\ , \leq)$ order field.

Remark:

$R^+ = \{x \in R\ ; x > 0\}$

$R^- = \{x \in R\ ; x < 0\}$

Propositions: Let $(R, +, .)$ be a field, then prove the following

1. $\forall a, b, c \in R\ , if\ a + b = b + c\ , then\ a = c$
2. $\forall a, b, c \in R\ , if\ a.b = c.b\ \ , then\ a = c$

3. $\forall a, b \in R$,prove that:

1. $-(-a) = a$
2. $(a^{-1})^{-1} = a$
3. $(-a) + (-b) = -(a + b)$
4. $(-a).b = -a, b$
5. $if\ a.b = 0$ then either $a = 0\ or\ b = 0$

Proof (5):

Let $a \neq 0$, T.P $b = 0$

Since $a \neq 0$, then $\exists a^{-1} \in R\ s.t\ \ a.a^{-1} = 1$

$a^{-1}(a.b) = 0$

$(a^{-1}.a).b = 0$

$1.b = 0 \rightarrow b = 0$

Let $b \neq 0$, T.P $a = 0$

Since $b \neq 0$, then $\exists b^{-1} \in R\ s.t\ \ b.b^{-1} = 1$

$(a.b)b^{-1} = 0$

$a.(b.b^{-1}) = 0$

$a.1 = 0 \rightarrow a = 0$

Absolute Value:

let $a \in R$, the absolute value of a is:

$$|a| = \begin{cases} a & if\ a > 0 \\ 0 & if\ a = 0 \\ -a & if\ a < 0 \end{cases}$$

$|a|: R \rightarrow R^{+} \cup \{0\}$ is the function of absolute value.

Properties of absolute value.

Theorem: let a be a real number, then

1. $|x| < a\ \leftrightarrow -a < x < a$
2. $|X| > a\ \leftrightarrow x > a\ or\ x < -a$

Corollary: let $a \in R^{+}$ and $b \in R$, then

1. $|x - b \leq a\ iff\ b - a \leq x \leq b + a$
2. $|x - b| \geq a\ iff\ x \geq b + a\ \ or\ \ x \leq b - a$

Let $a, b \in R$ and k be areal number, then

1. $|a| \geq 0$
2. $|a| = 0$ iff $a = 0$
3. $a^2 = |a|^2$
4. $|ab| = |a|.|b|$
5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
6. $|ka| = |k|.|a|$

Example: $\forall a \in R, \sqrt{a^2} = |a|$

Proof:

If $a > 0$ then $\sqrt{a^2} = a$

If $a < 0$ then $\sqrt{a^2} = -a$

by def absolute value to a we have

$$|a| = \begin{cases} a = \sqrt{a^2} \;\; if \;\; a \geq 0 \\ -a = \sqrt{a^2} \;\; if \;\; a < 0 \end{cases}$$

وفي كلتا الحالتين يكون لدينا $|a| = \sqrt{a^2}$

The triangle inequality

Theorem: if $a, b \in R$, then $|a + b| \leq |a| + |b$

Proof:

$$\begin{aligned} |a + b|^2 = (a + b)^2 &\leq a^2 + 2ab + b^2 \\ &\leq |a|^2 + 2|ab| + |b|^2 \\ &\leq (|a| + |b|)^2 \end{aligned}$$

$\therefore |a + b| \leq |a| + |b|$

Corollary: if $a, b \in R$, then $|a - b| \geq |a| - |b|$

Def: let $S \subset R$ S is said to be bounded above if there is some real numbers m s.t $x \leq m$ $\forall x \in S$, m is called upper bounded of S

LCH (3)

Proposition:
If $\emptyset \neq S \subset R$ and $\sup(S) = M$, then $\forall p < M \ \exists x \in S$ s.t $p < x \leq M$
i.e.: if $\sup(S) = M$ then $\forall \epsilon > 0$, $\exists x \in S$ s.t $M - \epsilon < x \leq M$
proof:

let $\sup(S) = M$ then $\forall x \in S,\ x \leq M$
T.P $\forall x \in S$, $p < x$?
Suppose that $x \leq p$, $\forall x \in S$
$\rightarrow$ p is upper bounded for S, but by hypothesis $p < M = \sup(S)$ ………. C!
$\therefore\ \exists\ x \in S \ni p < x \leq M$.

Theorem: The set N of natural numbers is unbounded above in R
Proof:

Suppose N is bounded above.
By completeness axiom
N has a supreme M
Let $\sup(N) = M$
From proposition above $\exists n \in N$ s.t $M - 1 < n < M$.
Then $M - 1 < n \rightarrow M < n + 1$,
But $n + 1 \in N$
And $n + 1 > M = \sup(N) \rightarrow C!$
Therefore, N is unbounded above

Theorem: Archimedan property

If $x \in R^{++}$ then for any $y \in R$, there exists $n \in N$ s.t $n > y$

Def: let F a field, F is called Archimedean filed, if for any $x \in F$, $\exists n \ \in N$ s.t $n > x$
i.e.: N is abounded above in F
Ex:

1. R is Archimedean field
2. Q is Archimedean field

3. $s = \{a + b\sqrt{2} : a, b \in Q\}$ is Archimedean field

Theorem: Denseness property

Between any two distinct reals, there exists infinitely many rationales and irrationals

LCH (4)

Def: (irrational numbers Q')

Let Q' be a complement of Q in the real number R.

i.e.: $Q' = R - Q$, we called is set of irrational numbers

remark: $R = Q \cup Q'$

Theorem: prove that $\sqrt{2}$ is irrational number

i.e.: There are no rational numbers whose square is 2

i.e.: $\nexists\ x \in Q \ni x^2 = 2$

proof:

suppose $\sqrt{2}$ is rational number i.e. $\sqrt{2} = \frac{m}{n}$

So $2 = \frac{m^2}{n^2}$, then $m^2 = 2n^2$

Case 1:

m and n are odd.

Since m is odd $\to m^2$ is odd

Since n is odd $\to n^2$ is odd

But $2n^2$ is even $\to\ m^2 = 2n^2 \to C!$

Case 2:

m is even and n is odd, then $m = 2p$

and $m^2 = 4p^2$,$\to 4p^2 = 2n^2 \to 2p^2 = n^2 \to C!$

Case 3:

m is odd and n is even, then, since m is odd

$\to m^2$ is odd, and $2n^2$ is even $\to m^2 = 2n^2 \to C!$

$\therefore\ \sqrt{2}$ is irrational number

Theorem: Q is not Complete field

Theorem: for every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$

i.e.:$\forall x > 0, \forall n \in N, \exists!, y \in R^+ \; s.t \; y = \sqrt[n]{x}$

Theorem: if $\frac{m}{n}$ and $\frac{p}{q}$ are rationales and $q \neq 0$ then $\frac{m}{n} + \sqrt{2}\frac{p}{q}$ is irrational number

Proof:

Suppose $\frac{m}{n} + \sqrt{2}\,\frac{p}{q}$ is rational

Then there is $r, s \in Z, s \neq 0 \; s.t \; \frac{m}{n} + \sqrt{2}\frac{p}{q} = \frac{r}{s}$

So $\sqrt{2}\frac{p}{q} = \frac{r}{s} - \frac{m}{n} \to \sqrt{2} = \frac{p}{q}\left(\frac{rn-sm}{sn}\right) \in Q$

So $2 = \left(\frac{q(nr-sm)}{psn}\right)^2 \to !$ with theorem: $\nexists\, x \in Q \ni x^2 = 2$

Theorem: Between any two distinct rationales there is an irrational number.

LCH (5)

Ex:

1. Prove $x^2 \geq 0, \forall x \in R$
2. Let a, b be tow real s.t $a \leq b + \epsilon \; \forall \epsilon > 0$ then $a \leq b$

   Proof (2):

   Suppose $a > b$

   Then $a + a > b + a$

   $\frac{2a}{2} > \frac{b+a}{2}$

   $a > \frac{b+a}{2}$ ………….(1)

   Take $\epsilon = \frac{a-b}{2} > 0$ (Since $a > b$, then $a - b > 0 \to \frac{a-b}{2} > 0$)

   $a \leq b + \epsilon \to a \leq b + \frac{a-b}{2} = \frac{2b+a-b}{2} = \frac{a+b}{2} < a$

   From (1) ……………. C!

   $a \leq b$

Ex:

1. Q is order field $(A_1 \to A_{14})$
2. C is field but not order

   since: if $x = 1 \to x = \sqrt{1} \to x^2 = -1 < 0 \to C!$

   since: $(x^2 \geq 0, \forall x \in R)$

Metric space

Def: let X be anon-empty set and $d: X \times X \to R^+$ be a mapping. We say that order (X, d) is metric space if it is satisfying the following:

1. $d(x, d) \geq 0 , \forall x, y \in X$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$
4. $d(x, y) = 0 \leftrightarrow x = y$

Not: d is called metric mapping
$d(x, y)$ is a distance between x and y

Remark: A mapping $d: X \times X \to R^+$ is called a pseudo metric for X iff d satisfies (1,2,3) in the above definition and $d(x, x) = 0 , \forall x \in X$

Cauchy - Shwarz inequality
Let $a = (a_1, a_2 , \dots , a_n)\ and\ b = (b_1 , b_2 , \dots b_n)$ be two tripe of complex number , then:

$$\sum_{i=1}^{n} |a_i + b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} . \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}$$

Minkowskis inequality

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} \quad , p \geq 1$$

Ex: if $X = R\ and\ d(x, y) = |x - y|$, show that (X,d) is a metric space.
Solution:

1. $d(x, y) = |x - y| \geq 0$ by def. of Absolute value
2. $d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x)$
3. $d(x, z) = |x - z| = |x - y + y - z|$
   $\leq |x - y| + |y - z|$
   $= d(x, y) + d(y, z)$
4. $d(x, y) = 0$ iff $x = y$
   $d(x, y) = 0$ iff $|x - y| = 0$
   iff $x - y = 0$
   iff $x = y$

$\therefore (X,d)$ is a metric space

Discrete metric space

Let $X \neq \emptyset$ and $d: X \times X \to R$ s.t

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

$\forall x, y$, show that (X,d) is metric space

Solution:

1. $d(x,y) \geq 0$, $\forall x, y \in X$ (by def. d)
2. $d(x,y) = d(y,x)$?
   if $x = y \to d(x,y) = 0 = d(x,y)$
   if $x \neq y \to d(x,y) = 1 = d(y,x)$
3. Let $x, y, z \in X$ T.P $d(x,y) \leq d(x,y) + d(y,z)$?
   if $x = z$ then $d(x,z) = 0$
   since $d(x,y) \geq 0$ and $d(y,z) \geq 0$ then
   $$d(x,z) \leq d(x,y) + d(y,z)$$
   if $x \neq z$ then $d(x,z) = 0$
   since $d(x,z) = 1$ and either $x \neq y\ or\ x \neq z$, $y = z$
   either: $d(x,z) = d(x,y) = d(y,z) = 1$
   or: $d(x,z) = d(x,y) = 1$ and $d(y,z) = 0$
   then: $d(x,z) \leq d(x,y) + d(y,z)$
   $1 \leq 1 + 1$
   $1 \leq 1 + 0$

LCH (6)

Ex: show that (X,d) is pseudo metric space but not metric where

$d: X \times X \to R$, $d(x,y) = |x^2 - y^2|$, $forall\ x, y \in R$.

Solution:

Let $x, y, z, \in R$

1- $d(x,y) = |x^2 - y^2| \geq 0$, by def Abs. Value

2- $d(x,y) = |x^2 - y^2| = |-(y^2 - x^2)| = |y^2 - x^2| = d(y,x)$

3- $d(x,y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2| \leq |x^2 - z^2| + |z^2 - y^2|$
$\leq d(x,z) + d(z,y)$

4- $d(x,x) = |x^2 - x^2| = 0 , \forall x \in R$

$\therefore (X,d)$ pseudo metric space but not metric space,

since, if $d(x,y) = 0 \rightarrow |x^2 - y^2| = 0 \rightarrow x^2 - y^2 = 0 \rightarrow x^2 = y^2$

$\rightarrow x = y$

ex: let $x = 1 , y = -1$

then $d(x,y) = d(1,-1) = |\ 1^2 - (-1)^2| = 0$, but $1 \neq -1$

Def: let (X,d) be a metric space $S,T \subseteq X , p \in S$ then

1- The distance between p and S is
   $d(p,S) = \inf\{d(p,x) : x \in S\}$
2- The distance between S and T is
   $d(S,T) = \inf\{d(x,y) : x \in S , y \in T\}$
3- Diameter of S is $d(S) = \sup\{d(x,y) : x,y \in S\}$
4- S is called bounded, if $\exists\ M \in R^{++}$, s.t $d(x,y) \leq M , \forall x,y \in S$.

Def: let (X,d) be a metric space and $S \subseteq X$, S is called open set, if $\forall x \in S , \exists r > 0$ s.t $B(x,r) \subset S$

Ex: if $S = \emptyset$, then S is open set

If $x \in S \rightarrow \exists r > 0\ s.t\ B(x,r) \subset S$

$F \rightarrow F\ or\ T : T$

LCH (7)

If S = X , then S is open set

Solution:

Since all balls is contains in X

Any open interval is open set. But the convers is not true

Solution:

Let $x \in s \rightarrow x \in (a,b) \subseteq (a,b) = S.$

So. S is open set

Ex: Let $S = (-1,1) \cup (2,3)$

Let $x \in s$, then $x \in (-1,1)\ or\ x \in (2,3)$

Then $x \in (-1,1) \subset S\ or\ x \in (2,3) \subset S$

$\therefore$ S is open set. But is not open interval

Any ball is open set.

Proof:

$\forall y \in B(x,r), \exists w > 0, s.t\ B(y,w) \subset B(x,r)$?

Let $w = r - d(x,y) > 0$

Let $Z \in B(y,w) \longrightarrow d(z,y) < w$

$d(Z,y) \leq d(x,y) + d(y,z)$

$\leq d(x,y + w$

$= d(x,y) + r - d(x,y)$

$= r$

Then $Z \ \in B(x,r) \longrightarrow B(y,w) \subset B(x,r)$

This is true for all y in B(x,r)

So B(x,r) is open set

$\boldsymbol{S = \{x\}, x \ \in R}$ **is not open set**

Since there is not open interval in S Containing x and Contained in S

i.e (($\forall\, r\ > 0, \exists B(x,r) = (x - r, x + r) \subset S$))

$\boldsymbol{[a,b], [a,b), [a,\infty)\ and\ (-\infty, b]}$ **are not open set**

Proof:

If S=[a,b] ,then S is not open set ?

Since, if $x = a\ \rightarrow\ \forall r > 0\ , B(a,r) = (a - r, a + r) \not\subset [a,b]$

The intersection of any tow open set is open set

i.e ((the intersection of any finite family of open set is open))

Proof:

Let $A = \{\, S_k :\ S_k\ is\ open\ set\ k = 1,2, \dots, n\}$

$T.p\ \bigcap_{k=1}^{n} S_k$ is open set

Let $x \in \bigcap_{k=1}^{n} S_k \rightarrow x \in S_k, \forall k$, but S_k is open set $\forall k$, then $\exists r_k > 0$ s.t

$B(x, r_k) \subset S_k$

Let $r = \min\{r_1\ , r_2\ , \dots, r_n\}$

Then $B(x,r)\ \subset S_k, \forall k.$

$\therefore B(x,r) \subset \bigcap_{k=1}^{n} S_k$, therefore $\bigcap_{k=1}^{\infty} S_k$ is open set.

Theorem: the infinite intersection of open sets is not necessary open set.

Ex: let $S_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ $\forall x \in R$, open interval.

$n = 1 \rightarrow s_1 = (x - 1, x + 1)$

$n = 2 \rightarrow S_2 = (x - \frac{1}{2}, x + \frac{1}{2})$

$n = 3 \rightarrow S_3 = (x - \frac{1}{3}, x + \frac{1}{3})$

. .

. .

When $n \rightarrow \infty \bigcap_{k=1}^{\infty} S_k = \{x\}$ is not open

Theorem: the union of any family (finite or infinite) – (countable or uncountable) of open set is open

Proof:

Let $A = \{S_\alpha , S_\alpha \; is \; open \; set \; \alpha \in \Lambda\}$

T.P: $\bigcup_{\alpha \in \Lambda} S_\alpha$ is open set

Let $x \in \bigcup_{\alpha \in \Lambda} S_\alpha \rightarrow \exists \alpha \in \Lambda \; s.t \; x \in S_\alpha$

Since S_α is open set $\rightarrow \exists \alpha > 0 \; s.t$

$B(x, r_\alpha) \subset S_\alpha$, then $x \in B(x, r_\alpha) \subset S_\alpha \subset \bigcup_{\alpha \in \Lambda} S_\alpha$

This is true $\forall x \in \bigcup_{\alpha \in \Lambda} S_\alpha$, therefore $\bigcup_{\alpha \in \Lambda} S_\alpha$ is open set

Theorem: S is open iff S is the Union of balls

LCH (8)

Def: let X be anon-empty set and τ is a family of subsets of X, if τ satisfy the following

1- $\phi , X \in \tau$
2- If $G , H \in \tau \rightarrow G \cap H \in \tau$
3- If $\{G_\lambda\} \in \tau \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \in \tau$

Then, the order pair (X, τ) is called topological Space.

Theorem: every metric space is topological space.

Proof:

Let (X, d) be a metric space and $\tau =$ the family of all open subsets of X, then

1- ϕ , X open sets $\rightarrow \phi, X \in \tau$
2- $G_1 , G_2 \in \tau \rightarrow G_1, G_2$ are open sets
$\rightarrow G_1 \cap G_2 \in \tau$
3- If $G_\lambda \in \tau , \lambda \in \Lambda \rightarrow \forall \lambda , G_\lambda$ open subset of X

$\rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda$ open set of
$\rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda \in \tau$
$\therefore (X,\tau)$ is a topological space

Def: let $d_1$and d_2 be two metric mapping in the set X, then d_1, d_2 are called Equivalent if every open set in (X, d_1) is open in (X, d_2) and Vice Versa

Def: let (X, d) be a metric space and $S \subseteq X$, S is called closed set if S^c is open Set where $S^c = X - s$ (Complement of S)

Ex:

1- $S = X$ is closed set.
Solution:
Since $S^c = X^c = \phi$ open set
2- $S = \phi$ is closed set
Solution:
since $S^c = \phi^c = X$ is open set
3- $S = [a,b], [a,b), S = (-\infty, b]$ are closed set in R
Solution:
if $S = [a,b] \rightarrow S^c = (-\infty, a) \cup (b, \infty)$ open set $\rightarrow S$ is closed set
4- In R , $S = \{x\}$ is closed set
Since :
$S^c = (-\infty, x) \cup (x, \infty) \rightarrow S^c$ is open, So S is closed set.
5- Any finite set in R is closed set
Solution:
let $S = \{x_1, x_2, \dots, x_n\} \subseteq R$.
$S^c = (-\infty, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$
So, S^c is open , then S is closed set
6- If $S = N$, $S = Z$, then S is Closed set
Solution:
let $S = N$
then $S^c = (-\infty, 1) \cup (1,2) \cup (2,3) \dots (\bigcup_{n=4}^{\infty}(n, n+1))$
$\rightarrow S^c$ is open $\rightarrow$ S is closed
if $S = Z \rightarrow S^c = (\bigcup_{n=1}^{\infty}(-(n+1), -n)) \cup (-1,0) \cup (0,1) \cup (\bigcup_{n=1}^{\infty}(n, n+1))$
S^c is open, then S is closed

7- The Union of finite number of closed sets is closed.
Solution:
let $A = \{S_i, ;\ S_i\ closed\ set\ in\ X\ , i = 1,2, \dots , n\}$
T.P: $\bigcup_{i=1}^{n} S_i$ is closed set
i.e. T.P $(\bigcup_{i=1}^{n} S_i)^c$ is open set
Since S_i is closed, $\forall i$ then S_i^c is open $\forall i$
and $\bigcap_{i=1}^{n} S_i^c$ is open
So, $(\bigcup_{i=1}^{n} S_i)^c$ is open $\quad ((\bigcup_{i=1}^{n} S_i)^c = \bigcap_{i=1}^{n} S_i^c)$
therefore $\bigcup_{i=1}^{n} S_i$ is closed.

Remark: the infinite union of closed sets is not necessary closed set

Ex: let $S_n = \left\{ \left[\frac{-n}{n+1}, \frac{n}{n+1}\right] : n \in N \right\}$, S_n is closed interval, Is $\bigcup_{n=1}^{\infty} S_n$ is closed?
Solution:
If $n = 1 \to S_1 = \left[\frac{-1}{2}, , \frac{1}{2}\right]$
If $n = 2 \to S_2 = \left[\frac{-2}{3}, \frac{2}{3}\right]$
. .
. .
When $n \to \infty \Longrightarrow \lim_{n\to\infty} \frac{\pm n}{n+1} = \lim_{n\to\infty} \frac{\pm\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \pm 1$
$\therefore \bigcup_{n=1}^{\infty} s_n = (-1,1)$ open set

Theorem: The infinite intersection of closed set S is closed?

Def: let X be a metric space and $S \subseteq X, p \in X$, p is called an accumulation point of S if every open set contain p , contains another point q s.t $p \neq q , q \in S$.
i.e.: p is a cc. point of S if $\forall U$, U is open set $p \in U$, then $U - P \cap S \neq \phi$

Remark: Since every open set is Union balls. So, we can define acc. Point as following:
P is acc. Point of S, if $\forall r > 0\ B(p,r) - \{p\} \cap S \neq \phi$

* S' is the closure of all acc. Point of S (Derived set)

* $\overline{S}$ is the closure of S and $\overline{S} = S \cup S'$
* P is not acc. Point, if $\exists U$, U is open and $p \in U$
  S.t $U - \{p\} \cap S = \phi$. (i.e. $\exists r > 0$, $B(r,p) - \{\, p \,\} \cap S = \phi$

Ex: let $s = \{1,5\}$, find S' and $\overline{S}$

Solution: TO find S' there are some cases

LCH (10)

$x = 1$, $x = 5$, $x < 1$, $x > 5$, $1 < x < 5$

If $x = 1 \rightarrow$ x is not acc. Point since , $\exists r > 0$

$B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$B(1,1) - \{1\} \cap \{1,5\} = (0,2) - \{1\} \cap [1,5\} = \emptyset$

If $x = 5 \rightarrow$ x is not acc. Point, since $\exists r > 0$, $B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$\rightarrow B(5,1) - \{5\} \cap \{1,5\} = (4,6) - \{5\} \cap \{1,5\} = \emptyset$

If $x < 1 \rightarrow$ x are not acc. Point since $x \in (x-1,1)$ and $(x-1,1) \cap S = \emptyset$

If $x > 5 \rightarrow x$ are not acc. Point, since $x \in (5, x+1)$ and $(5, x+1) \cap S = \emptyset$

If $1 < x < 5$ are not acc. Point since, $x \in (1,5)$ and $(1,5) \cap S = \emptyset$

So, S has no a acc. Point then $S' = \emptyset$ and $\overline{S} = S \cup S' = S \cup \emptyset = S$.

Let $s = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \right\} = \left\{\frac{1}{n}, n = 1,2,3, \dots . \right\}$ show that $S' = \{0\}$

If $S = (a,b)$, find S'

Solution:

If $x = a \rightarrow x$ is acc. Point since $\forall r > 0$,

$a \in B(0,r) = (a-r, a+r)$ and $B(a,r) - \{a\} \cap S \neq \emptyset$

If $x = b \rightarrow x$ is acc. Point, since $\forall r > 0$, $b \in B(b,r)$

$B(b,r) = (b-r, b+r)$ and $B(b,r) - \{b\} \cap (a,b) \neq \emptyset$

If $a < x < b \rightarrow x$ are acc. Point since $\forall\, r > 0$,

$x \in B(x,r) = (x-r, x+r)$ and $B(x,r) - \{x\} \cap S \neq \emptyset$

That is $(x-r, x+r) - \{x\} \cap (a,b) \neq \emptyset$

If $x < a \rightarrow x$ are not acc. Point since $x \in (x-1, a)$ and $(x-1, a) \cap S = \emptyset$

If $x > b \rightarrow x$ are not acc. Point, since $x \in (b, x+1)$ and $(b, x+1) \cap (a,b) = \emptyset$

$\therefore S' = [a,b] \rightarrow \overline{S} = S \cup S' = [a,b]$

LCH (11)

Def: A sub set A of a metric space X is said to be dense if $\overline{A} = X$

Ex: prove that $\bar{Q} = R$ (i.e., Q dense set in R)

Solution:

If $x \in R$, then x is acc. Point in Q.

Since any open interval Contain x Contains infinitely rational and irrationals

Then $Q' = R$

So $\overline{Q} = Q \cup Q' = Q \cup R = R$

Def: a metric space is called separable if it has a countable dense subset.

Ex: R separable since Q countable and $Q \subseteq R$, with Q dense in R

Theorem: let X be a metric space, $S \subseteq X$ then

1- S is closed iff $S' \subset X$

2- $\overline{S}$ is closed set

3- $\overline{S} = S$ iff S closed set

4- $\overline{S}$ is smallest closed set contains S.

Compact Space

Def: let (X, d) be a metric space, $\emptyset \neq S \subseteq X$, if the set $\{U_\lambda : U_\lambda \text{ open set}, \lambda \in \Lambda\}$ is a family of open subsets of X such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$, then the family $\{U_\lambda\}$ is called open cover for S in X.

- If the family $\{U_\lambda\}$ is finite and $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ then $\{U_\lambda\}$ is called finite cover.
- Let $\{U_\lambda\}$ and $\{U_\alpha\}$ be to open cover for S and $U_\lambda \in \{U_\alpha\}\ \forall \lambda$, then $\{U_\lambda\}$ is called subcover for $\{U_\alpha\}$

Def: let A be a subset of a metric space (X, d), A is called compact set if every open cover for A in X has a finite subcover.

LCH (12)

Exp: Any finite subset B of matric space (X, d) is **compact set**

Ex: R is not compact

.

Ex : Any open interval A=(a,b) is not compact

Ex : Any closed interval A=[a,b] is Compact.

Proof :

Since we can restrict any open cover for A to finite subcover such as :

Let $\epsilon > 0, B = \{(a-\epsilon, a+\epsilon, (a,b), (b-\epsilon, b+\epsilon)\}$

(a) (b]

Theorem: ((Bolzano weir strass theorem))

In compact space X, every infinite subset S of X has at least one accumulation point.

Theorem : In compact metric space, every closed subset is compact.

Proof : X be a compact metric space, and A be a closed subset of X, then A^c is open. T.P A is compact.

Let $B = \{U_\lambda : U_\lambda \text{ is open set in } X, \forall \lambda \in {}^\wedge \}$ be open cover for A.

Then $A \subseteq \bigcup_{\lambda \in {}^\wedge} U_\lambda$

Sine $X = A \cup A^c \subseteq (\bigcup_{\lambda \in {}^\wedge} U_\lambda) \cup A^c$,

But A^c is open set then $\bigcup_{\lambda \in {}^\wedge} U_\lambda \cup A^c$ is open cover for X, since X is compact set , then there exists a finite member $\lambda_1, \lambda_2, \ldots \ldots, \lambda_n$ such that

$$X = A^c \cup \left(\bigcup_{i=1}^{n} U_{\lambda i} \right)$$

Since that $X = A^c \cup (\bigcup_{i=1}^{n} U_{\lambda i})$ .Since $A \cap A^c = \emptyset$, then $A \subseteq \bigcup_{i=1}^{n} U_{\lambda i}$

$\Rightarrow$ B has a finite subcover $\{ U_{\lambda 1}, U_{\lambda 2}, \ldots \ldots, U_{\lambda n}\}$. For A, $\Rightarrow$ A is compact.

LCH (13)

Theorem: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is closed

Theorem: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is bounded

Remark: In metric space

Compact $\rightarrow$ Closed + bounded

$\not\leftarrow$

Theorem: Let $\{I_n : n = 1,2,3, \dots\}$ be a family of closed interval if $I_{n+1} \subset I_n, \forall n, then\ \bigcap_{n=1}^{\infty} I_n = \emptyset$

Theorem: (**Hien-Bord Theorem**)
Every closed and bounded subset of R^n, $n \geq 1$, is compact.

Chapter Three

Sequences in Metric Space

Definition: Let S be any set a function f whose domain is the set N and the range is S is Called a sequence in S.

i.e. $f: N \to S$, where $\forall\, n \in N, \exists\, x_n \in S\ s.t\ f(n) = x_n$

1. $< \frac{1}{5n} > = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \dots$
2. $< \frac{1}{n+1} > = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$
3. $< 4 > = 4, 4, 4, \dots$
4. $< n - 3 > = -2, -1, 0, 1, \dots$

Def: Let (X, d) be a metric space and $< X_n >$ be seq. in X, then $< X_n >$ is said to be converges to appoint in X, if $\forall\, \epsilon > 0, \exists k \in N\ s.t\ d?(X_n, x) < \epsilon, \forall n > k$. We write $X_n \to x$ or $\lim_{n \to \infty} X_n = x$, x is called

LCH (14)

A Limit point of $< X_n >$.

If $\forall n > K$, does not Converge, them $< X_n >$ is called divergent Sequence.

Not that: K depend on ϵ only.

التغير الهندسي للتعريف التقارب

$$(X_n \to x)$$

يعني الكرة التي مركزها x ونصف قطرها ϵ تمتلك عدد غير منتهي من حدود او نقاط المتتابعة X_n لانه

$\forall \epsilon > 0, \exists\, k \in N\ s.t\ d(X_n, x) < \epsilon, \forall n > k \implies X_n \in B(x, \epsilon)$.

Ex: Let $< X_n > = < 1 >$ constant seq. show that $\lim_{n \to \infty} X_n = 1$

$< 1 >$ convergs to 1 since $\forall\, \epsilon > 0, \exists\, k \in N$

$s.t\ d(X_n, x) = |1 - 1| = 0 < \epsilon, \forall n > k$

Ex: Let $< X_n >$ be a seq. defined by $X_n = \begin{cases} n \ if \ n \leq 50 \\ \\ 3 \ if \ n \geq 50 \end{cases}$ .show that $\lim_{n\to\infty} X_n = 3$

Solution:

$< X_n \geq 1,2,3, \dots, 50 \,, 3 \,, 3 \,, 3 \,, \dots$

$\forall \epsilon > 0 \,, \exists k = 50 \ s.t \ d(X, x) = |3 - 3| = 0 < \epsilon$

Ex: Show that $\lim_{n\to} X_n = 2$, where $< X_n > = < \frac{2n-3}{n+1} >$

Solution:

$\forall \ \epsilon > 0$, to find $K \in N$ s.t $d(X_n, x) < \ \epsilon, \forall n > k$?

$$d(X_n, x) = \left|\frac{2n-3}{n+1} - 2\right| = \left|\frac{2n-3-2(n+1)}{n+1}\right| = \left|\frac{2n-3-2n-2}{n+1}\right|$$

$$= \left|\frac{-5}{n+1}\right| = \frac{5}{n+1}$$

$\forall \epsilon > 0$, by Arch. Property $\to \exists K \in N \ni$

$\forall k > 5 \to \frac{5}{\epsilon} < k.$

$\forall n > K \to n + 1 > k + 1 \ and \ k + 1 > k \,, k > \frac{5}{\epsilon}$

$\Rightarrow n + 1 > k + 1 > k > \frac{5}{\epsilon}$

$\frac{1}{n+1} < \frac{\epsilon}{5} \,, \forall n > k$

Exc:

1. Let $< X_n > =< \frac{2}{\sqrt{n}} >$, show that $\lim_{n\to\infty} X_n = 0$
2. Let $< X_n > = < \frac{5n-4}{2-3n} >$, show that $\lim_{n\to\infty} X_n = -\frac{5}{3}$
3. Let $< X_n > = < \frac{2-7n}{1-5n} >$, show that $\lim_{n\to\infty} X_n = \frac{7}{5}$

Show that the following sequence are divergent

1. $< X_n > = < \sqrt{n} >$
2. $< X_n > = < (-1)^n >$
3. $< X_n > 3^n >$
4. $< X_n > = < \frac{n^2}{2n-1} >$

Theorem: If $< X_n >$ is convergent sequence in (X, d), then $< X_n >$ has a unique limit point.

Proof:

Suppose $< X_n >$ has two limit points x and y with $x \neq y$ and $d(x,y) = \epsilon$

Since $X_n \to y \Longrightarrow \forall \epsilon > 0, \exists k_2 \in N\ s,t\ d(x,y) < \frac{\epsilon}{2}$

Let $k = \max\{k_1, k_2\}$

Since $d(x,y) \leq d(x,x_n) + d(x_n,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow d(x,y) < \epsilon, \forall \epsilon > 0$

This true only when $d(x,y) = 0 \Rightarrow x = y \to C!$

$\therefore < X_n >$ has a unique limit point.

LCH (15)

Definition: A seq. $< X_n >$ is called bounded the set $\{X_n : n \in N\}$ is bounded

i.e. $< x_n >$ is bounded if $\exists m > 0\ s.t\ d(x_n, x_m) \leq M, \forall n, \forall m.$

Ex:

1. $< \frac{(-1)^{n+1}}{n} > = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$

   $|x_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \leq 1 \Longrightarrow < x_n >$ is bounded

   and $M = 1$

2. $< 5 + \frac{(-1)^{n+1}}{n} > = 6, \frac{9}{2}, \frac{16}{3}, \ldots$

   $< x_n \geq 5 + \frac{1}{n} \leq 5 + 1 = 6 \Longrightarrow < x_n >$ is bounded

   and $M = 6$

3. $< n + (-1)^n > = \begin{cases} < n-1 >, & if\ n\ is\ odd \\ < n+1 >, & if\ n\ is\ even \end{cases}$

4. $|x_n| = \begin{cases} |n-1| \geq 0 \\ |n+1| \geq 2 \end{cases}$

Theorem: In metric space. Every convergent sequence is bounded.

Proof:

Let $< x_n >$ be a convergent sequence in (X,d) and $x_n \to x$, to prove $< x_n >$ is bounded

Since $x_n \to x \implies \forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x) < \epsilon, \forall n > k$
That $\epsilon = 1 \implies d(x_n, x) < 1, \forall n \in k$.
Let $r = \max\{1, d(x_1, x), d(x_2, x), \dots, d(x_n, x)\}$
$\implies d(x_n, x) < r$
$\therefore < x_n >$ is bounded and $M = 2r$

Remark: The convers of above theorem is not true.

Ex: $< (-1)^n > = -1, 1, -1, 1, \dots$
$|x_n| = |(-1)^n| = 1 \implies < x_n >$ is bounded and $M = 1$
$< (-1)^n >$ is divergent?

Remake: If $< x_n >$ unbounded, then $< x_n >$ is divergent.

Proof:
Suppose that $< x_n >$ converged and unbounded sequence.
Since $< x_n >$ Convergent $\to < x_n >$ bounded by theorem (In metric space, every conv. Seq. is bounded) $\to$ C! ,So $< x_n >$ unbounded is $< x_n >$ is divergent

Ex:

- $< x_n > = < \sqrt{n-1} > = 0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \dots$ unbounded $\implies < x_n >$ divergent
- $< x_n > = < n^2 - n > = 0, 2, 6, 11, \dots$ unbounded $\implies < x_n >$ divergent

LCH (16)

Definition: Let $< x_n >$ be a real sequence. Then it is called

- Non – decreasing. If $x_{n+1} \geq x_n, \forall n$
- Non – increasing. If $x_{n+1} \leq x_n, \forall n$.
- Not monotone. If it does not increasing and decreasing.

Ex:

* $< x_n > = < \frac{1}{\sqrt{n}} >$

$x_n = \frac{1}{\sqrt{n}}, x_{n+1} = \frac{1}{\sqrt{n+1}}$

$\forall n, n+1 > n \implies \sqrt{n+1} > \sqrt{n} \to \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{2}} \to x_{n+1} \leq x_n$

$\therefore < x_n >$ is non – increasing

* $< x_n > = < \frac{n}{n+1} >$

  $x_n = \frac{n}{n+1}$, $x_{n+1} = \frac{n+1}{n+2}$

  $x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)-n(n+2)}{(n+1)(n+2)} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$

  $\therefore x_{n+1} - x_n > 0 \rightarrow x_{n+1} > x_n , \forall n , \therefore < x_n >$ non – decreasing
* $< x_n > = < (-1)^n >$ not monotone
* $< x_n > = < \frac{(-1)^n}{\sin(n)} >$ not monotone.
* $< x_n > = < (-5)^n >$ not monotone.

Theorem: Every monotone bounded real seq. is convergent

Ex: $< x_n > = \langle \frac{(-1)^n}{n} \rangle > 0$

$< x_n >$ Convergent seq. but not monotone.

Ex: Show that $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Theorem: Let (X, d) be a metric space and $S \subseteq X$:

i. If $< x_n >$ seq. in S and $x_n \rightarrow x$ then $x \in S$ or $x \in S'$
ii. If $x \in S \ or \ x \in S'$, then there exists a sequence $< x_n >$ in S s.t $x_n \rightarrow x$

Definition: The sequence $< x_n >$ is a sub sequence of $< x_n >$, if $< m >$ is increasing sequence in N.

Ex: find a sub Seq. of the following seq.

1. $< x_n > = < \sqrt{n} >$

   Solution:

   $< \sqrt{n} > = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$

LEC (17)

Let $< m > = < 2n >$ increasing Seq. in N, the Sequence is

$$< Xm > = < \sqrt{2n} > = \sqrt{2}, \sqrt{4}, \sqrt{6}, \ldots$$

Let $< m > = < n + 3 >$ increasing seq in N, the sub seq is

$$< m > = < \sqrt{n+3} > = \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$$

Theorem: Let $< x_n >$ be a convergent Seq and $\lim_{n\to\infty} X_n = x$ then the sub seq $< X_{nm} >$ also conv. To x, where $n \to \infty$

Proof:

Since $x_n \to x, \forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x) < \epsilon, \forall n > k$

Choose $nr > k$, then $\forall m > r \to nm > nr > k$

$\Longrightarrow d(x_{nm}, x) < \epsilon, \forall nm > k$

$\Longrightarrow < x_{nm} > \to x.$

Definition: Let (X, d) be a metrices space and $< x_n >$ be a seq. in X we say that $< x_n >$ is a principle. (Caushy) seq. if $\forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x_m) < \epsilon, \forall n, m > k.$

Ex: prove that $< \frac{1}{n} >$ is Caushy seq in R?

Solution: $\forall\ \epsilon > 0$, to find $k \in N$ s.t $d(x_n, x_m) < \epsilon, \forall n, m > k, \forall n, m > k.$

Let $m > n \to d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

Since $\epsilon > 0$ (by Arch. Prop) $\to\ \exists k \in N$ s.t

$k\epsilon > 2 \to \frac{2}{k} < \epsilon$

$\forall\ n > k, d(x_n, x_m) = |x_n - x_m| < \frac{2}{n} < \epsilon, \forall n, m > k \to < X_n >$ is Caushy seq.

Theorem: I metric space (X, d), every Convergent seq. is Caushy.

Remark: The Converse of the above theorem. Is not true by the following example.

Ex: Let $X = IR^{++}$ positive numbers $d(x, y) = |x - y|, \forall x, y \in R^{++}, \forall n > k.$

$< x_n > = < \frac{1}{n} >$ is Caushy seq.

But $\frac{1}{n} \to 0 \notin R^{++}$

$\therefore < \frac{1}{n} >$ is not Conv

Theorem: In metric Space (x, d) every Caushy seq. is bounded.

Ex: Let $< x_n > = (-1)^n$ be a seq.

$< x_n >$ is bounded seq, but not Caushy Seq

Since $d(-1,1) = 1 < \epsilon, \forall \epsilon > 0$
If $\epsilon = \frac{1}{2} \to 2 < \frac{1}{2} \to C!$

Theorem: For any real number $r, \exists$ rational Caushy Seq $< x_n >$ Conv to r.

LEC (18)

Definition: Let(X, d) be a metric space we say that X is Compete. If every Cauchy Seq. In X coverage to a point in X.
i.e.: X is complete. If $\forall < X_n >$ Cauchy Seq. $\to \exists \bar{x} \in X\ s.t\ X_n \to X$.

Theorem: Cantor's theorem for Nested sets.

Proof:

Let (X, d) be a Complete matric Space and $< E_n >$ be a seq of closed bounded Subset of X such that $E_1 \supset E_2 \supset \cdots E_n \supset E_{n+1}\ \forall n$ and the Sequence of Positive numbers $< \ daim\ E_n > \to 0$, then $\cap E_n =$ Singleton point

Remark: The condition of closed sets of Cantor's theorem is necessary.

Ex: Let $E_n = \left(0, \frac{1}{n}\right)$ be the open intervals, $E_{n+1} \subset E_n$, and $daim(E_n) = \frac{1}{n} \to 0, \forall n$
E_n is bounded and not closed. Prove that $\cap E_n = \emptyset$

Proof:

Suppose $\cap E_n \neq \emptyset \to \ \exists r \in E_n\ s.t$
$r \in \left(0, \frac{1}{n}\right), \forall n$
Since $r > 0$, by Arch.pvop ,$\exists k \in N\ s.t$
$kr > 1 \to \frac{1}{k} < r \to C!$
➔ $\cap E_n = \emptyset$

Corollary: Let $< \pm n >$ be aseq of closed intervals, $I_n = [a_n, b_n]$ such that

1. $I_n \supset I_{n+1}$
2. $\lim_{n\to\infty} |I_n| = 0,$ then $\cap I_n =$singleton Point

Theorem: R^n is Complete metric Space, $n \geq 1$

i.e.: (Every Cauchy sequence in R^n is Convergent)

Theorem: Let $< X_n >, < Y_n >$ and $< Z_n >$ real Sequence s.t $\forall n, X_n \leq Y_n \leq Z_n$ and $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Z_n = a$ then $\lim_{n\to\infty} Y_n = a$

Theorem: let $< X_n >$ be a real sequence such that $< X_n >$ Converge to 0 and $X_n \geq 0, p > 0$ then $< X_n^p >$ converges to 0

Proof:

$< X_n^p > = x_1^p, x_2^P, x_3^p, \ldots$

Since $< X_n > \to 0 \to \forall \epsilon > 0, \exists\, k \in N\ s.t$

$|X_{n-0}| = |X_n| < \epsilon^p, \forall n > k$ and

$|X_n . X_n \ldots . X_n| = |X_n||X_n| \ldots . |X_n| = |X_n|^p < \left(\epsilon^{\frac{1}{p}}\right)^p, \forall n > k$

$< X_n^p > \to 0$.

Chapter four

Infinite Series

Def: Let $\langle x_n \rangle$ be a real seq the series of the form, if $x_1 + x_2$ then itis Called infinite series, and it is written as $\sum_{n=1}^{\infty} x_n$.

If the series of the form $x_1 + x_2 + \cdots + x_n$, then itis Called finite Series.and written as $\sum_{k=1}^{n} x_k$

Def: Let $\sum_{n=1}^{\infty}$ an be a finite series, the seq $\langle S_n \rangle$ is called the sequence of Partial sums of $\sum_{n=1}^{\infty}$ a_n

where $S_1 = a_1$

$$S_2 = a_1 + a_2$$
$$S_3 = a_1 - a_2 + a_3$$
$$\vdots$$
$$S_n = a_{1+}a_2 + \cdots + a_n$$

Def:
let $\sum_{n=1}^{\infty} a_n$ be infinite series, then it is said toble

1. Converge, if $\langle S_n \rangle$ converge
2. diverge, if $\langle s_n \rangle$ diverge.
3. If $\langle$ Sn $\rangle$ Converge to b. then $\sum_{n=1}^{\infty} a_n = S_n$.

Example:
let $a_n = 1, \forall n$, then

$\sum_{n=1}^{\infty} a_n = 1 + 1 + 1 + \cdots$

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 1 = 2$

$S_3 = a_1 + a_2 + a_3 = 1 + 1 + 1 = 3$

.

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$S_n = a_1 + a_2 + a_3 + \cdots + a_n = 1 + 1 + 1 + \cdots + 1 = n$
The seq of partial sum is $\langle S_n \rangle = \langle n \rangle$ is divergent Since it is unbounded
$\Rightarrow \sum_{n=1}^{5}$ an is diverge.

Example: let $\sum_{n=1}^{\infty} a_n = 3 - 3 + 3 - 3 + \cdots$

$$S_1 = a_1 = 3$$
$$S_2 = a_1 + a_2 = 3 - 3 = 0$$
$$S_3 = \mathrm{a}_1 + \mathrm{a}_2 + a_3 = 3 - 3 + 3 = 3$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n = \begin{cases} 3 & , if\ n\ odd \\ 0 & , if\ n\ even \end{cases}$$

The Sequence of partial Sum $\langle S_n \rangle$ is divergent

$\therefore$ $\sum_{n=1}^{\infty} a_n$ is divergent

Example:

Let: $\sum_{n}^{\infty} a_n = 2 + 4 + 2 + 4 + 24$

$n = 1$

$S_1 = a_1 = 2$

$S_2 = a_1 + a_2 = 2 + 4 = 6$

$S_3 = a_1 + a_2 + a_3 = 2 + 4 + 2 = 8$

$Sn = a_1 + a_2 + a_3 + a_4 = 2 + 4 + 2 + 4 + \cdots = ?$

The sequence of partial sums $\langle S_n \rangle$ is unbownded, then $\langle s_n \rangle$ is diverg ent so $\sum_{n=1}^{\infty} a_n$ is diverge

Exercises

let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Harmonic Series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \text{ divergent}$$

proof:
$S_1 = a_1 = 1$
$s_2 = a_1 + a_2 = 1 + \frac{1}{2}$
$S_3 = a + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3}$
$S_n = a_1 + a_2 + \cdots + a_{n-1} + \frac{1}{2} + \cdots + \frac{1}{n}$
$S_{n+1} = a_1 + a_2 + \cdots + a_n - a_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}$
$S_{n+n} = \frac{1}{2n} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$
let $m = 2n$

$$\begin{aligned}(S_m - S_n) \quad &= \left|\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right| \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} \\ &= n \cdot \frac{1}{2n} = \frac{1}{2}\end{aligned}$$

If $\epsilon = \frac{1}{2}$, then $|S_m - S_n| > \epsilon$
$\therefore \langle S_n \rangle$ is not Caushy sequence $\Rightarrow \langle S_n \rangle$ is not Convergent.
So $\sum_{n=1}^{2} a_n$ is diverge.

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$
where a $> 0, r$ is called the base of Series. the sequence of partial, Sum is

$$sn = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

(1) if $|r| = 1$
$\therefore S_n = a + a^{-1} + a + \cdots + a = n \cdot a$
$\langle S_n \rangle = \langle n_a \rangle$ diverg $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ diverge.

(2) if $|r| > 1$

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$

$\rightarrow S_n - rS_n = a - ar^n$

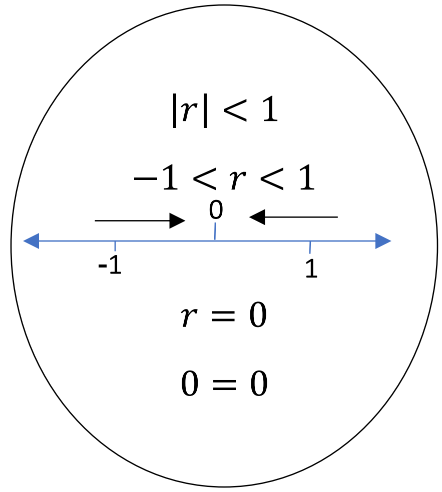
$S_n(1-r) = a(1-r^n)$

$\therefore S_n = \frac{a(1-r^n)}{(1-r)}$

When $n \rightarrow \infty \Rightarrow \lim_{n\rightarrow\infty} S_n = \lim \frac{a(1-r^n)}{(1-r)}$

$= \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

$\therefore \sum ar^{n-1} = \frac{a}{1-r}$ .Converge



(3) if $|r| > 1$

$$s_n = \frac{a(1-r^n)}{1-r}$$

when $n \longrightarrow \infty; r^n = \mp\infty \Rightarrow S_n \rightarrow \infty$

$\therefore$ Sn diverge.

$\therefore \sum_{n=1}^{\infty} ar^{n-1}$ diverge.

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{divenge} & \text{if } |\text{r}| > 1 \\ \text{Ganverge} & \text{if } |\text{r}| < 1 \end{cases}$$

$= \sum_{n=1}^{\infty} ar^{n-1} - \frac{a}{1-r}$

Example

Tet $\sum_{n=1}^{\infty} a_n = 1 + \frac{5}{2} + \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^3 + \cdots$ Geometric series.

$$a = 1, v = \frac{5}{2} \Rightarrow |r| = \left|\frac{5}{2}\right| = \frac{5}{2} > 1$$

$\therefore \sum_{n=1}^{\infty}$ an divenge:

$\sum_{\sum=1}^{\infty} a_{n=1} = 1 - \frac{3}{4} + \frac{9}{16} \ \frac{27}{64} + \cdots$ Geometric Series.

$$\sum_{n=1}^{\infty} ar^{n-1} = 1 + \left(\frac{-3}{4}\right) + \left(\frac{-3}{4}\right)^2 + \left(\frac{-3}{4}\right)^3 + \cdots$$

$a = 1, v = \frac{3}{4} \Rightarrow$

$|r| = \left|-\frac{3}{4}\right| = \frac{3}{4} < 1$

$\therefore \sum_{n=1}^{\infty} a_n$ is Converger and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{15}{1+\frac{3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

Theorem:
If $\sum_{n=1}^{\infty}$ an Convergent, them $\lim_{n\to\infty} a_n = 0$
(that is, $\forall \in > 0, \exists k \in N, s.t |a_n - 0| \langle \epsilon, y_n \rangle k$

proof
Suppose $S_n = a_1 + a_2 + \cdots + a_n$
$\sum_{n=1}^{\infty}$ an canvergent, then $\langle S_n \rangle$ Convergent

$\Rightarrow \langle S_n \rangle$ canshy sequence.
$\therefore \forall \in > 0, \exists k \in N$, sit $|S_m - s_n| < E, \forall n, m > k$
let $m = n + 1$
So $|S_m - S_n| < \epsilon \to |S_{n+1} - S_n| = |a_{n+1}| < \epsilon, \forall > k$
$\to |a_n| < \epsilon, \forall n > k, So\ |a_n - 0| < \epsilon, \forall n > k.$
then $\lim_{n\to\infty} a_n = 0$

Example:

$\langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle \to 0$

and $\lim_{n\to\infty} a_n = 0$ bul $\sum_{n=1}^{\infty} \mathrm{a_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diveroe

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Corollary:
If $\lim_{n\to\infty} a_n \neq 0$ diverge.

proof:
Suppose that $\sum_{n=1}^{\infty} \mathrm{a_n}$ Convergent.

then, $\lim_{n\to\infty} a_n = 0$, by theorem, $\longrightarrow$ C!

Example

$\sum_{n}^{\infty} a_n = \sum (\sqrt{m} - \sqrt{n-1})$

$\sum_{n=1}^{\infty}$ a_n Diverge, but $\lim_{n\to\infty} a_n = 0$

Exercises

(1) $\sum_{n^3}^{\infty} \frac{1}{\sqrt{n}}$

(2) $\sum_{n=1}^{a} \sqrt{\frac{n}{3n+5}}$

(3) $\sum_{n=1}^{\infty} \frac{n^3+2}{2n(n+5)}$

Theorem

If $\sum_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty}$ bn are Convergant Series and $k \in R$, then

(1) $\sum_{n=1}^{\infty}(\text{an} + b_n)$ convergent and $\sum_{n=1}^{\infty}(\text{an} + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(2) $\sum_{n=1}^{m_n} ka_n$ convergent and $\sum_{n=1}^{\infty} ka_n = k\sum_{ns}^{\infty} a_n$

proof: (1)

let $\langle s_n \rangle$ be a sequence of partial sums of

$\sum_{n=1}^{\infty} a_n$ and

$\langle t_n \rangle$ bea seq of partipl sam of $\sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty}$ an convergent, so $\exists s \in R$ s.t $\sum_{n=1} = s$

and $\langle Sn \rangle \longrightarrow S \Rightarrow \text{Lim}_{n\to\infty} S_n = S$.

also, $\sum_{n=1}^{\infty} b_n$ Canergent, then $\exists t = R$, s.t $\sum_{n=1}^{\infty} b_n = t$ and $\langle t_n \rangle \to t \ \to \text{Lim}_{n\to\infty} t_n = t$

$\text{Lim}_{n\to\infty}(s_n + t_n) \to S + t$, but $\langle s_n + t_n \rangle$ is the seq of partial sum of $\sum_{n=1}^{\infty}(\text{an} + b_n) \to$

$\sum_{n=1}^{\infty} a_{n+} b_n = \sum_{n+1}^{\infty} a_{n+} \sum_{n=1} b_n = s + t$

(2) let $\langle s_n \rangle$ be a seq of partial sums of $\sum_{n=1}^{\infty} a_n$ but $\sum_{n=1}^{\infty} a_n$ Convergent $\Rightarrow \exists s \in \mathbb{R}$ s,t

$\sum_{n=1}^{\infty} a_n = s$ and $\langle S_n \rangle = S$, $\lim_{n\to\infty} s_n = S$.

, $\lim_{n\to\infty} k\, s_n = k$, $\lim_{n\to\infty} s_n = kS \to \langle k\, S_n \rangle \to k_s$

then $\sum_{ns_1}^{2} ka_n = kS = k\sum_{n\leq 1}^{\infty}$

then $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$

Exercises

(1) Given an example for two divergent Sories but their Sum is Convergent Series.

Sol:

let $\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \frac{1}{n}$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} -\frac{1}{\mathrm{n}}$

the $\sum_{ns1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} 0 = 0$ con and $\langle a_n + b_n \rangle \longrightarrow 0$

Series test اختبار المتسلسلات

(1) Comparison test:

Theorem: If $0 \leq a_n \leq b_n \ \forall\, n \in N$, then

(1) $\sum_{n=1}^{\infty}$ bn convergent, then $\sum_{n=1}^{\infty}$ an Canvergont

(2) $\sum_{n=1}^{\infty} a_n$ divergent, then $\sum_{n\leq 1}^{\infty}$ bn divergent

`of partial sums of $\sum_{n_1 1}^{\infty} b_n$ sina $0 \leqslant$ an $\leqslant b_n$, then

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$\quad \leqslant b_1 + b_2 + b_3 + \cdots + b_n$

$= \mathrm{t_n}$

but $\sum_{n=1}^{\infty} b_n$ Convergent, thei $\langle t_n \rangle \Rightarrow$ t as $n \to \infty$ $b_n \geqslant 0 \Rightarrow \langle\, t_n \,\rangle$ increasing seq and $t_n \leq t\,, \forall_n$ and $S_n \leqslant t_n, \forall n$, the $S_n \leqslant t$, $\mathrm{S_0}$ $\langle s_n \rangle$ is bounded

$\to \langle S_n \rangle$ is bounded and increasing (mono ton) $\Rightarrow \langle \mathrm{s_n} \rangle$ Convergent sequence

$\Rightarrow \sum_{ns}^{\infty}$ an Cowlergint.

(2) Suppose $\sum_{n=1}^{\infty}$ bn Converyont

by (1), $\sum_{n=1}^{\infty}$ an Convergent and $\to C!$, so $\sum_{n,1}^{\infty} b_n$ divergent

P – series

$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0.$

$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^{\mathrm{p}}} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Canerge} & \text{if } p > 1 \\ \text{diveng} & \text{if } < p \leqslant 1 \end{cases}$

Examples

(0) $\sum_{n=1}^{\infty} \frac{1}{5n^3} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n^3}, p = 3 > 1$

then P - Series $\rightarrow p = 3\rangle 1$, sa Caniergent

(2) $\sum_{n} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{t}}}, p^{-1} \frac{1}{2} < 1$

Then p series, $P = \frac{1}{2} < 1, d$ inevgent

Theuron.

let Lan and $\sum b_n$ be posifive term Series s.t $\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0$

then

Example:

D) $\sum^{\infty} n^3 - 1$

$n = 04n^5 - 3n^2 + 3$

$a_n = \frac{n^3-1}{4a^5-3n^2+3} \geqslant 0$, choose bns $\frac{1}{n^2}$ to Compave

$\sum$ thans $\sum \frac{1}{n^2}$ Convergant (p-sories $p = 2 > 1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1\text{in} \frac{n^3 - 1}{4n5 - 3n^2 + 3} \div \frac{1}{n^2}$$

$$= \lim_{n\to\infty} = \frac{n^5 - n^2}{4n^5 - 3n^2 + 3} = \lim_{n\to\infty} \frac{\frac{n^5}{n5} - \frac{n^2}{n^5}}{4\frac{n^5}{n^5} - 3\frac{n^2}{n^5} + \frac{3}{n^5}}$$

$$= L_{n\to\infty} \frac{1 - \frac{1}{n^3}}{4 - \frac{3}{n^3} + \frac{3}{n^5}} = \text{Lim}_{n\to\infty} \frac{1}{4} = \frac{1}{4} = 0$$

by theoramabove $\sum_{n=0}^{\infty}$ an Convargant.

$\Big(2\sum_{n=0}^{\infty} \frac{2n + 1}{n^2 + 2n + 1}$

(3) Ratiotest) a.mil sl: in. 1 - If $b < 1 \Rightarrow \sum$ an Cowergent.

2 if $h > 1 \Rightarrow 2$ an divergent.

3 -if $b = 1 \Longrightarrow$ no infurmations.

Examples

(1) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(2) $\sum_{n<0}^{\infty} \frac{n}{3^n}, \cdots$ · Convergant.

(3) $\sum_{n\leq 0}^{2} n^2$

let $a_n = n^2, a_n + (n+1)^2$

$L_{n\to\infty}\dfrac{a_n+1}{a_n} = 1\text{mim}_{n\to\infty}\dfrac{(n+1)^2}{n^2} = \dfrac{1,m\dfrac{n^2+2n+1}{n^2}}{}$

$\text{lim}_{n\to\infty}\dfrac{\dfrac{n^2}{n^2}+\dfrac{2n}{n^2}+\dfrac{1}{n^2}}{\dfrac{n^2}{n2}} = \text{Lim}_{n\to\infty}\dfrac{1+\dfrac{2}{n}+\dfrac{1}{n}2}{=}1$

$\therefore b = 1, bw + \sum_{n=1}^{\infty} n^2$ is divergent

$(4)\sum_{ni0}^{\infty}\dfrac{1}{n^2}$

lel ans $\frac{1}{n^2}, -$ avit $= \frac{1}{(n+1)^2}$

$= \text{Lim}_{n^2+2n+1} = \text{lim}_{n\to\infty}\left(\dfrac{1}{1+0}\right) = \text{Lim}_{n\to\infty} |= 1$

So $b = 1$

but $\sum\frac{1}{n^2}$ is Canventi, 5 ina $p = 2 > 1$

Theorem

tet $\sum_{n+1}^{m}$ be a series, $ax > 0, \forall n$, if $\exists b \in R$ sit

$n\sqrt{a_n} = b$

1 - if $b < 1 \Rightarrow$ \suman Comergent

2 - if $b > 1 \rightarrow$ enp di

3 if be1 $\Rightarrow$ no is Por matios.

Examples: Is the Pollowing Sarics Gurmergent?

(1) $\sum\frac{5n}{2(3)^n}$

let $a_{in} = \frac{5n}{2(3)} > 0$

$\text{lim}_n\sqrt{\frac{5n}{2(3)^n}} = L\cdot m - \sqrt{\frac{5}{2}}\cdot\frac{\sqrt[n]{n}}{\sqrt[3]{3^n}} = \text{Lim}_n\sqrt{\frac{5}{2}}\cdot\frac{\sqrt{n}}{3} = 1\cdot\frac{1}{3} = \frac{1}{3}$

$$= 1\cdot\frac{1}{3} = \frac{1}{3}$$

$$b = \frac{1}{3}\Big\langle 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ Convevgent.}$$

$$\text{Excersices: } \sum_{ns0}^{n} 2^2$$

Defintion The number e

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Remark

The Series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is Convergent Serice.

prosp $s_n = 1 + \frac{1}{1!} + \frac{1}{2t} + \frac{1}{sT} + \cdots + \frac{l}{n!}$

$= 1 + 1 + \frac{1}{2 \times 1} + \frac{1}{3 \times 2 \times 1} + \frac{1}{4 \times 3 \times 2x_1} + \frac{1}{n!}$

$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{(n-1)}}$

$= 1 + 1 + \frac{1}{2} + \frac{1}{t^2} + \frac{1}{2^3} + \frac{11 + -1}{x^{n-1}}$

$$\frac{1/2}{1/2} = 1 \Rightarrow \text{ sn } \langle t + 1 + 1 = 3$$

$\therefore S_n \langle 3 \Rightarrow \langle S_n \rangle$ boumided and incveasing $\Rightarrow \langle \text{sn} \rangle$ Canvevge

Excersices

prave that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$

Example

prove theat e is irpational Aumber 05

Suppase e is ralional number $\Rightarrow 3m, n > 0$ sit

$e = \frac{m}{n}.$

$\because e = \sum_{n=0}^{\infty} \frac{1}{n!} \Rightarrow S_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \infty + \frac{1}{n!}$

$e - 5n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!}$

$= \frac{1}{(n+1)!} + \frac{1}{(n+2)(n+1)!} + \frac{1}{(n+3)(n+2)(n+1)!} +$

$= \frac{1x}{(n+1)!}\left[i + \frac{1}{(n+2)} + \frac{1}{(n+3)(n+2)} + \infty\right]$

$< \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} +\right]$

$= \frac{1}{(n+1)!}, \frac{n+1}{n} = \frac{1}{n+1)n!} \cdot \frac{n+1}{n} = \frac{1}{n \cdot n!}$

$(n!)e \in N$ since $nl_1 = n! \frac{m}{n} = n(n-1)! \frac{m}{n}$

$$= (n-1)!\, m \in N$$

eand n!s $n = n! \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$

$$= n! + n! + \frac{n1}{2!} + \frac{n!}{3!} + \cdots - 1$$

Since $n \geqslant 1 \Rightarrow 3$ natwal number $(e - 5n)n!$
sil $0 < e - 5_n < \frac{1}{n} < 1$ by (1) $-C!$
e is inpational amariber

- Alternating Series aj 23 4 a al

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - 44 +$$
$$\text{or } _{n=1}(-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

Theoren (Alternating Series test)
The series $\sum_{n=1}^{\infty} (-2)^{n-1} a_n +$ is Convergent if
(1) $a_n > 0, v_n$
(e) $a_{n+1} \leqslant 0, vn$
(3) Lim an $= 0$
Example Is the fallewing shries are Comergent.
(1) $\sum_{n \leqslant 1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$a_n = \frac{1}{n} > 0, an + 1 = \frac{1}{n+1} < \frac{1}{x} = a_n,\ \lim_{n \to \infty} \frac{1}{n} = 0 = \sum_n^n \frac{(-1)^n}{n}$ Gonvergent.
8
$\Big(2\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$?
Absolule and Comditional Convergencen)
Pejintion (Absetutely Covergant)
A series $\sum$ an is Called absolutely convergena is the associated series $\sum$ lanl Garvergent.
Defintion (Conditionally Convergent).
A series $\sum a_n$ is guled Condilionally Convergenl if the ssociated series $\sum_a$ Covengentbut $\sum$ lanl divengent
(1) lel $\sum a_n = \sum_{n=0}^{\infty} \frac{(-1)^4}{2^n}$
$\Rightarrow \Sigma$lan$| = \sum_{n=0}^{\infty} | \frac{(-y)^n}{2^n} | = \sum_{n=0}^{\infty} \frac{1}{2^n}$, Geometric saries

(2) let $\sum ax = \sum_{n=0}^{\infty} \frac{(-1)^2}{n+1}$
i $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ not alosolutely Cowengent).
$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

$a_n = \frac{l}{n+1}, a_{n_7 1} = \frac{1}{n+2} < \frac{1}{n+1} = \text{cm}$

$\therefore \sum \frac{\sum(-1)^n}{n+1}$ Conditiontity convergent

Thearem

$\Rightarrow \langle 5n \rangle$ is Cauthy seq.

If $\langle t_n)$ is aseq if partial sions of \suman

$\Rightarrow t_n = a_1 + a_2 + \cdots + a_n$ and

$\Rightarrow \langle$ tn $>$ canchy ser.

$$\Rightarrow \langle t_n \rangle \text{ convergent } \Rightarrow \text{ 2an Convengant}$$

If Eang 5 bn Cowvergant series.

Is $\sum a_n \cdot \sum b_n - (a_n + a_n + c) \cdot (l_1 + b_i +$

$= a_1(b_1 + b_t + y) + a_2(b_1 + b_{2x}) + \cdots$

Camergent?

Definitin(Cavshy produch of Series)

let $\sum_{n=1}^{\infty}, \sum_{n}^{\infty} b_n$ be twa Suries and $C_{n=}\sum_{ks}^{n} a_k b_{n-k} = a_{10} bn + a_1 b_{n-1} + \cdots + a_n b_0$

Exanple

$\sum_{n=0}^{\infty} a_n + \sum_{n>0}^{\infty} b_n$ not Convergent

$$= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right)$$

(power Series

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + ax + a_2 x^2 + 93x^3 + \cdots$$

where $x \in R$ is Called power scriesin x

Exc shew that \sumx+rak $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is Convergant

وفي الختام نسأل الله التوفيق

اللهم قني عذابك

يوم تبعث عبادك